Invariants of a semi-direct sum of Lie algebras

2004 J. Phys. A: Math. Gen. 37 5635

(http://iopscience.iop.org/0305-4470/37/21/009)

J. Phys. A: Math. Gen. **37** (2004) 5635–5647 PII: S0305-4470(04)76065-5

Invariants of a semi-direct sum of Lie algebras

J C Ndogmo

Department of Mathematics, University of Western Cape, Private Bag X17, Bellville 7535, South Africa

E-mail: jndogmo@uwc.ac.za

Received 13 February 2004, in final form 2 April 2004
Published 12 May 2004
Online at stacks.iop.org/JPhysA/37/5635
DOI: 10.1088/0305-4470/37/21/009

Abstract
We show that any semi-direct sum L of Lie algebras with Levi factor S must be perfect if the representation associated with it does not possess a copy of the trivial representation. As a consequence, all invariant functions of L must be Casimir operators. When $S = \mathfrak{sl}(2, \mathbb{K})$, the number of invariants is given for all possible dimensions of L. Replacing the traditional method of solving the system of determining PDEs by the equivalent problem of solving a system of total differential equations, the invariants are found for all dimensions of the radical up to 5. An analysis of the results obtained is made, and this leads to a theorem on invariants of Lie algebras depending only on the elements of certain subalgebras.

PACS number: 02.20.Sv

1. Introduction

The subject of invariant functions of the coadjoint representation of Lie algebras has given rise to many publications in recent scientific literature [1–5]. These are functions on the dual space L^* of the Lie algebra L that are invariant under the coadjoint action of the connected Lie group G generated by L.

The invariants of physical symmetry groups provide the quantum numbers needed in the classification of elementary particles. Thus, by making use of the eigenvalues of the Casimir operators of the Poincaré group, Wigner achieved a classification of particles according to their mass and spin [6]. One of the most significant applications of the invariant functions in physics is in the theory of dynamical symmetries, in which the Hamiltonian is written in terms of the Casimir operators of the corresponding Lie symmetry group and its subalgebras [7].

For semisimple Lie algebras, these functions are well known, following a paper by Racah [8], which was mainly a continuation of a work by Casimir [9] and other physicists. For this class of Lie algebras, there always exists a fundamental set of invariants that consists of homogeneous polynomials, and the degree of transcendence of the associative algebra generated by these functions over the base field $\mathbb{K}$ of characteristic zero equals the dimension of the Cartan subalgebra.

The Levi decomposition theorem gives a preliminary classification of Lie algebras into semisimple and solvable ones, and a third class consisting of semi-direct sums of semisimple and solvable Lie algebras. One fact that has emerged about these types of semi-direct sums of Lie algebras is that all the invariants that have been computed for them in the recent literature are polynomials [5, 10, 3]. In this paper, we consider a semi-direct sum of Lie algebras of the form $L = S \oplus_\pi \mathcal{R}$, where S is semisimple and $\mathcal{R}$ is the solvable radical, and where π is a representation of S in $\mathcal{R}$ defining the $[S, \mathcal{R}]$-type commutation relations. Unless otherwise stated, we shall refer to a Lie algebra of this form simply as a semi-direct sum of Lie algebras. The base field $\mathbb{K}$ is assumed to be of characteristic zero. In section 3, we show that if π does not possess a copy of the trivial representation, then L must be perfect, that is, equal to its derived subalgebra. As a consequence of a result of [1], this implies that all invariants of L must be polynomials. Other important consequences of this result are discussed. It is noted in particular that all irreducible representations π satisfy the stated criteria.

In the next section, we move on to tackle the problem of the explicit determination of the number of invariants of L with Levi factor $S = \mathfrak{sl}(2, \mathbb{K})$ and π an irreducible representation. Simple formulae giving this number are derived for all possible dimensions of $\mathcal{R}$. In section 5, we find it convenient to replace the most common method that consists in solving a system of first-order PDEs for the determination of the invariants by the equivalent problem of solving a system of total differential equations, as described in a book by Forsyth [11]. This approach provides a more tractable algorithm. The invariants are computed for all dimensions of the radical up to 5, and an analysis of the functions obtained is given. In particular, we show that when $\dim \mathcal{R} > 3$, all invariants depend only on the elements of $\mathcal{R}$. We show that this property always hold, with a rank condition, for Lie algebras L which are direct sums $L_1 \oplus L_2$ of subspaces, and where L_2 is an Abelian subalgebra. Some families of Lie algebras having this property are exhibited.

The study of the invariants of Lie algebras reduces to the case of indecomposable ones. We shall therefore assume, unless otherwise stated, that the Lie algebra L is indecomposable.

2. Invariant functions and Casimir operators

2.1. Formal invariants

Suppose that L is the finite-dimensional Lie algebra of the connected Lie group G, and denote by L^* the dual space of L. Let Ad: $G \to GL(L)$ and Ad*: $G \to GL(L^*)$ be the adjoint and the coadjoint representations of G, respectively. Thus for $g \in G$, $f \in L^*$ and $x \in L$ we have $g \cdot f(x) = f(\mathrm{Ad}_{g^{-1}}(x))$, where $g \cdot f$ stands for $\mathrm{Ad}_g^*(f)$. Denote by $C^\infty(L^*)$ the space of all analytic functions on L^*.

Definition 1. *A function $F \in C^\infty$ is called an invariant of the coadjoint representation if $F(g \cdot f) = F(f)$, for all $g \in G$ and $f \in L^*$.*

Let $\{e_1, \ldots, e_n\}$ be a basis of L, and $\{f_1, \ldots, f_n\}$ the corresponding dual basis in L^* given by $f_j(e_i) = \delta_i^j$. Let $\tilde{V}_i$ be the infinitesimal generator of the coadjoint representation associated with the basis vector e_i of L, and suppose that $x_1, \ldots, x_n$ is a coordinate system in L^* corresponding to the dual basis. It is a well-known fact [12] that the infinitesimal generators are given by

$$\tilde{V}_i = -\sum_{j,k} c_{ij}^k x_k \frac{\partial}{\partial x_j} \qquad \text{for} \quad i = 1, \ldots, n \tag{2.1}$$

where the c_{ij}^k are the structure constants of the n-dimensional Lie algebra L in the given basis. These vector fields on L^* form a Lie algebra that is homomorphic to L. We have more precisely, $[V_i, V_j] = \sum_k c_{ij}^k V_k$. On the other hand, by considering the infinitesimal action of Ad^* on L^*, it is easy to see [12] that a function $F \in C^\infty(L^*)$ is an invariant of the coadjoint representation if and only if it satisfies the system of partial differential equations

$$\tilde{V}_i \cdot F = 0 \qquad \text{for} \quad i = 1, \ldots, n. \tag{2.2}$$

The most common method for finding the invariants [2, 1, 4, 5, 13] is by solving the system of linear first-order partial differential equations given by (2.2). In contrast with the polynomial functions in terms of which the invariants of semisimple Lie algebras can always be expressed, the solutions to equation (2.2) generally involve various kinds of functions, including rational and logarithmic functions, as well as functions in arctan. They are therefore usually referred to as formal invariants of L. Because any functional relations among the invariants yield another invariant, they are determined by a maximal set of functionally independent invariants. Such a set is called a fundamental set of invariants. The number of invariants of L usually refers to the cardinality of this set. It is a well-known fact [13, 4] that the number $\mathcal{N}$ of invariants of the Lie algebra L is given by

$$\mathcal{N} = \dim L - \operatorname{rank}(\mathcal{M}_L) \tag{2.3}$$

where $\mathcal{M}_L = \left(\sum_{k=1}^n c_{ij}^k x_k\right)$ is the matrix of the commutator table of L.

2.2. *Invariant polynomial functions*

Let V be a finite-dimensional vector space. The symmetric algebra $\mathfrak{S}(V^*)$ is called the algebra of polynomial functions on V, and is often denoted [14] by $\mathfrak{P}(V)$. When a fixed basis $\{f_1, \ldots, f_n\}$ of V is given, $\mathfrak{P}(V)$ becomes identified with the algebra of polynomials in n variables, $f_1, \ldots, f_n$. Thus, the coadjoint representation Ad_g^* of the Lie group G acts naturally on $\mathfrak{P}(L)$, via $(\mathrm{Ad}_g^* f)(x) = f\big(\mathrm{Ad}_g^{-1}(x)\big)$, for $f \in \mathfrak{P}(L)$, and we see that the set denoted by $\mathfrak{P}(L)^I$ of all elements of $\mathfrak{P}(L)$ fixed by this action is precisely the algebra of polynomial invariants of the coadjoint representation.

Let $\mathfrak{A}(L)$ be the universal enveloping algebra of the Lie algebra L, and denote by $\mathfrak{A}(L)^I$ its centre. That is, $\mathfrak{A}(L)^I$ is the subset of elements commuting with all $x \in \mathfrak{A}(L)$, or equivalently, with all $x \in L$. The elements of $\mathfrak{A}(L)^I$ are called Casimir operators, and they commute with all elements of a representation. It follows by Schur's lemma that in any irreducible representation they are represented by scalars. Since any automorphism $\sigma : L \to L$ extends uniquely to an automorphism of $\mathfrak{A}(L)$, there is an action of $\mathrm{Ad}^*(G)$ on $\mathfrak{A}(L)$, and this sends $\mathfrak{A}(L)^I$ onto itself. It can be shown [14, 15] that $\mathfrak{A}(L)^I$ is precisely the set of all $\mathrm{Ad}^*(G)$-invariants of $\mathfrak{A}(L)$. Finally, it can be proved [1, 15, 8] that the associative algebras $\mathfrak{P}(L)^I$ and $\mathfrak{A}(L)^I$ are algebraically isomorphic. That is, there is a one-to-one correspondence between the polynomial invariants of the coadjoint representation and Casimir operators. When L is semisimple, the space of all invariants of the coadjoint representation is precisely $\mathfrak{P}(L)^I$. In this case, all invariant functions of the coadjoint representation are polynomials, and hence Casimir operators. In the next section, we show that this is also true for any nontrivial semi-direct sum $L = S \oplus_\pi \mathcal{R}$ of Lie algebras, when the representation π does not possess a copy of the trivial representation.

3. Properties of the invariant functions

Suppose that the finite-dimensional Lie algebra L over the field $\mathbb{K}$ of characteristic zero is a semi-direct sum of the semisimple Lie algebra S and the solvable ideal $\mathcal{R}$. That is, we have the vector space direct sum

$$L = S \dotplus \mathcal{R} \tag{3.1}$$

where $[S, S] = S$, $[\mathcal{R}, \mathcal{R}] \subset \mathcal{R}$ and $[S, \mathcal{R}] \subset \mathcal{R}$. Furthermore, we shall always assume that this semi-direct sum is nontrivial, which means that $[S, \mathcal{R}] \neq 0$ (otherwise L is decomposable). The given of the Lie algebra L is equivalent to the given of the semisimple Lie algebra S, the solvable algebra $\mathcal{R}$, and a representation π of S in $\mathcal{R}$, which defines the $[S, \mathcal{R}]$-type commutation relations. Furthermore, $\pi(z)$ must be a derivation of $\mathcal{R}$ for each $z \in S$. Indeed, the Jacobi identity applied to $z \in S$ and $u, v \in \mathcal{R}$ leads to

$$\pi(z) \cdot [u, v] = [\pi(z) \cdot u, v] + [u, \pi(z) \cdot v] \tag{3.2}$$

and justifies this assertion. Using equation (3.2) and the solvability of $\mathcal{R}$, it is easy to see that for any irreducible representation of S in $\mathcal{R}$, the radical $\mathcal{R}$ must be Abelian.

Let $\mathcal{R}^S = \{v \in \mathcal{R}\colon \pi(S)v = 0\}$. In [15], an element of the subspace $\mathcal{R}^S$ is called an invariant of the S-module $\mathcal{R}$. Denote also by $\pi(S)\mathcal{R}$ the subspace of $\mathcal{R}$ generated by all $\pi(s)\mathcal{R}$, where $s \in S$. Because S is semisimple, $\mathcal{R}$ is the direct sum of $\mathcal{R}^S$ and $\pi(S)\mathcal{R}$.

Lemma 1. *Let $L = S \oplus_\pi \mathcal{R}$ be a nontrivial semi-direct sum of the semisimple Lie algebra S and the solvable Lie algebra $\mathcal{R}$, and suppose that the representation π defines the $[S, \mathcal{R}]$-type commutation relations.*

(a) If π does not possess a copy of the trivial representation, then L is perfect, and it has therefore a fundamental set of invariants consisting of polynomials.
(b) The representation π does not possess a copy of the trivial representation if and only if $\pi(S)\mathcal{R} = \mathcal{R}$.

Proof. We first notice that we have in this case $\pi(S)\mathcal{R} = [S, \mathcal{R}]$. Now, for part (a), we see that π does not possess a copy of the trivial representation if and only if $\mathcal{R}^S = 0$. It then follows from the remark preceding the lemma that $[S, \mathcal{R}] = \mathcal{R}$, whence the equality $[L, L] = L$. The remaining part of the assertion is a consequence of a result of [1, corollary 2] asserting precisely that any perfect Lie algebra has a fundamental set of invariants consisting of polynomial functions.

For part (b), the result is a consequence of the fact that π does not possess a copy of the trivial representation if and only if $\mathcal{R}^S = 0$, and the equality $\mathcal{R} = \pi(S)\mathcal{R} \oplus \mathcal{R}^S$. □

Theorem 1. *Let $L = S \oplus_\pi \mathcal{R}$, with the usual notation, and suppose that the representation π is irreducible. Then L is a perfect Lie algebra, and has therefore a fundamental set of invariants that consists of polynomial functions.*

Proof. By part (a) of lemma 1, we only need to show that if π is irreducible, then it does not have a copy of the trivial representation. If π is irreducible, then since the semi-direct sum is nontrivial, the image space of a generic element $\pi(z)$ of the representation is clearly a nonzero invariant subspace, and its complementary subspace W is the largest subspace on which π acts trivially. By irreducibility $W = 0$, and thus π does not possess a copy of the trivial representation. □

Remark

(1) Not all semi-direct sums of Lie algebras are perfect. One such example is given by the 'optical Lie algebra' opt(2, 1) [16]. It is a seven-dimensional subalgebra of the de Sitter algebra $o(3, 2)$ and has the form $L = S \oplus_\pi \mathcal{R}$, where S is generated by $\{k_1, k_2, l_3\}$, the radical $\mathcal{R}$ is generated by $\{w, m, q, c\}$ and the commutation relations are given by

$$
\begin{array}{lll}
[w, m] = -[k_1, m] = \frac{1}{2}m & [k_2, q] = [l_3, m] = \frac{1}{2}m & [w, q] = \frac{1}{2}q \\
[k_1, q] = [k_2, m] = \frac{1}{2}q & [w, c] = -[m, q] = c & -[l_3, m] = \frac{1}{2}q \\
[k_1, k_2] = -l_3 & [k_1, l_3] = -k_2 & [k_2, l_3] = k_1.
\end{array}
$$

A simpler example is given by any Lie algebra of the form $L = S \oplus_\pi \mathcal{R}$, where the radical $\mathcal{R}$ is Abelian and $\mathcal{R}^S \neq 0$.

(2) For a Lie algebra of the form $L = S \oplus_\pi \mathcal{R}$, the condition that π does not possess a copy of the trivial representation (or equivalently, that $[S, \mathcal{R}] = \mathcal{R}$) is only a sufficient condition, but not a necessary condition, for L to be perfect. For example, the derived subalgebra of opt(2, 1) has c as a central element, and π has therefore a copy of the trivial representation. However, this subalgebra is perfect.

(3) Lemma 1 gives an interpretation of the sufficient condition $[S, \mathcal{R}] = \mathcal{R}$ for L to be perfect in terms of the representation π defining the $[S, \mathcal{R}]$-type commutation relations.

By the already cited result of Racah [8], any semisimple Lie algebra has a fundamental set of invariants consisting of polynomials. This result together with the Levi decomposition theorem and lemma 1 shows that there are only two types of Lie algebras that might not have a fundamental set consisting of polynomial invariants. The first type consists of semi-direct sums of Lie algebras of the form $L = S \oplus_\pi \mathcal{R}$, where the representation π possesses a copy of the trivial representation (and is therefore not irreducible). The second type consists of solvable non-nilpotent Lie algebras. Indeed by a result of [17], the invariants of nilpotent Lie algebras can all be chosen to be polynomials. However, invariants in a fundamental set for solvable non-nilpotent Lie algebras generally involve rational, logarithmic and other types of functions [4, 18, 10]. Although some non-nilpotent solvable Lie algebras having a fundamental set consisting of polynomials are given in [2], no characterization of such Lie algebras is available.

For semi-direct sums of Lie algebras, there is still no general result concerning the number of their invariants, contrary to the case of semisimple Lie algebras for which this number is known to be equal to the rank of the algebra [8, 14]. We derive this number for a particular Levi factor S in the next section.

4. The number of invariants of L

In this section we suppose that $S = \mathfrak{sl}(2, \mathbb{K})$ has the standard basis x, y, h in which the commutation relations are given by

$$[h, x] = 2x \qquad [h, y] = -2y \qquad [x, y] = h. \tag{4.1}$$

The following result is an immediate consequence of theorem 7.2 of [14] and theorem 13.11 of [19].

Theorem 2. *Let $S = \mathfrak{sl}(2, \mathbb{K})$, where $\mathbb{K}$ is a field of characteristic zero.*

(a) For any $m \in \mathbb{Z}^+$, there exists a unique (up to isomorphism) irreducible S-module of highest weight m.

(b) *For each value* $m \in \mathbb{Z}^+$ *of the highest weight, there exists a basis* $\{v_0, \ldots, v_m\}$ *of the* S*-irreducible module* $V = V(m)$ *in which the action of* S *is given by*

(b.1) $h \cdot v_i = (m - 2i)v_i$;

(b.2) $y \cdot v_i = (i + 1)v_{i+1}$;

(b.3) $x \cdot v_i = (m - i + 1)v_{i-1}$

where $i \geqslant 0$, v_0 *is the maximal vector, and where* $v_{-1} = v_{m+1} = 0$ *and* $v_i = (1/i!)y^i \cdot v_0$.

We shall also need the following result.

Lemma 2. *Let* $M = \left(\begin{smallmatrix} A & B \\ C & 0 \end{smallmatrix}\right)$ *be a matrix partitioned into four blocks of matrices* A, B, C *and* 0, *where* 0 *represents the zero matrix.*

(a) *If* A *is a non-singular square matrix of order* k, *then* M *has rank* k *if and only if* $CA^{-1}B = 0$.

(b) *If* M *is a non-singular square matrix of order* $2p$ *so that each of the block matrices is a square matrix of order* p, *then* M^{-1} *has the form* $M^{-1} = \left(\begin{smallmatrix} 0 & Y \\ Z & W \end{smallmatrix}\right)$, *where* Y, Z *and* W *are square matrices of order* p.

Proof. Part (a) is an immediate consequence of a theorem from a book by Gantmacher [20, p 47]. For part (b), we note that $M^{-1} = \left(\begin{smallmatrix} X & Y \\ Z & W \end{smallmatrix}\right)$ implies $ZB = I$ and $XB = 0$, where I is the identity matrix. Thus, B is invertible and hence $X = 0$. □

We now assume that, with the usual notation, $L = S \oplus_\pi \mathcal{R}$ has finite dimension n, and that $\dim \mathcal{R} = d$. Thus, L has dimension $n = d + 3$. We note that d may assume any positive value, by part (a) of theorem 2.

Theorem 3. *Let* $L = S \oplus_\pi \mathcal{R}$, *and set* $r = \operatorname{rank}(\mathcal{M}_L)$, *where* $\mathcal{M}_L$ *is the matrix of the commutator table of* L. *Then*

(a) $r = 2$, *for* $d = 1$.

(b) $r = 4$, *for* $d = 2, 3$.

(c) $r = 6$, *for* $d \geqslant 4$.

Proof. For part (a), we note that when $d = 1$, S acts trivially on the one-dimensional module $\mathcal{R}$, and thus $\operatorname{rank}(\mathcal{M}_L) = \operatorname{rank}(\mathcal{M}_S)$, where $\mathcal{M}_S$ is the matrix of the commutator table of S. It also follows from equation (4.1) that $\mathcal{M}_S$ can be put in the form

$$\mathcal{M}_S = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \qquad \text{with} \quad A = \begin{pmatrix} 0 & h \\ -h & 0 \end{pmatrix}$$

and for some block matrices B and C, we have $CA^{-1}B = 0$. The result then follows from part (a) of lemma 2.

For part (b), one can always write $\mathcal{M}_L$ in the form

$$\mathcal{M}_L = \begin{pmatrix} A_4 & B \\ C & 0 \end{pmatrix} \qquad \text{for} \quad d = 2, 3$$

where A_4 is a square non-singular matrix of order 4, and we have $CA_4^{-1}B = 0$. Thus, the result follows again from part (a) of lemma 2.

Finally, we note that

$$\mathcal{M}_L = \begin{pmatrix} M_6 & E \\ F & 0 \end{pmatrix} \qquad \text{for} \quad d \geqslant 4$$

where M_6 is a square matrix of order 6, of the form $\left(\begin{smallmatrix} A & B \\ C & 0 \end{smallmatrix}\right)$ and where A, B and C are 3×3 block matrices. Denoting by $m = d - 1$ the highest weight of the representation, we find that M_6 has determinant

$$-\big((-2+m)v_1\big((-1+m)v_1^2 - 3mv_0v_2\big) + 3m^2v_0^2v_3\big)^2$$

and this is different from 0 since it has $9m^4v_0^4v_3^2$ as a term. Thus, M_6^{-1} has the form $M_6^{-1} = \left(\begin{smallmatrix} 0_{3\times 3} & Y \\ Z & W \end{smallmatrix}\right)$, by part (b) of lemma 2. Noting now that the submatrices E and F have their last three rows and last three columns consisting of zeros, respectively, it follows that $M_6^{-1}E = 0$. By part (a) of lemma 2 again, we see that $\operatorname{rank}(\mathcal{M}_L) = \operatorname{rank}(M_6) = 6$. □

Theorem 4. *Suppose that the radical $\mathcal{R}$ has dimension d. Then the number $\mathcal{N}$ of invariants of $L = S \oplus_\pi \mathcal{R}$ is given by*

(a) $\mathcal{N} = 2$, for $d = 1$.
(b) $\mathcal{N} = 1, 2$, for $d = 2, 3$, respectively.
(c) $\mathcal{N} = d - 3$, for $d \geqslant 4$.

Proof. It is a direct consequence of equation (2.3) and theorem 3. □

5. Explicit determination

5.1. The method of total differential equations

The invariants are usually determined by solving the system of determining equations (2.2). This is a system of homogeneous linear first-order partial differential equations, and the method of characteristic is the most common for solving them. These equations are suitable for the determination of the invariants when the number of variables they involve is relatively low. They are widely used for the determination of the invariants [2, 4, 1, 5].

However, when the number of variables involved in the invariants becomes relatively high, the equivalent adjoint system of total differential equations becomes more appropriate for the determination of the invariants. It provides a more efficient algorithm involving a smaller number of change of variables and substitutions. In particular, the number of equations in the system corresponds to the number of invariants. Thus, only one total differential equation is to be solved when there is only one invariant, no matter what the initial number of equations in (2.2). We now derive the relationship between a system of integral equations and the corresponding adjoint system of total differential equations (see [11] for more details).

Let

$$\phi_j(u_1, \ldots, u_p, x_1, \ldots, x_q) = c_j \tag{5.1}$$

be a system of p integral equations in $p + q$ variables, where p and q are positive integers. Furthermore, assume that the functions ϕ_j are functionally independent, for $j = 1, \ldots, p$. Then, without loss of generality, this amounts to assuming that

$$\frac{\partial(\phi_1, \ldots, \phi_p)}{\partial(u_1, \ldots, u_p)} \neq 0. \tag{5.2}$$

Taking the differential in (5.1) yields

$$\sum_s \frac{\partial \phi}{\partial u_s}\, \mathrm{d}u_s + \sum_t \frac{\partial \phi}{\partial x_t}\, \mathrm{d}x_t = 0. \tag{5.3}$$

By the implicit function theorem, condition (5.2) implies that one can solve the system (5.1) for the variables u_j ($j = 1, \ldots, p$) in terms of the remaining q variables. Thus, we call the variables u_j the dependent variables, and the remaining q variables are the independent variables of (5.1). In particular, (5.2) implies that one can uniquely solve for the differentials $\mathrm{d}u_s$ in terms of the $\mathrm{d}x_t$ in (5.3). This yields the following system of total differential equations:

$$\mathrm{d}u_s = \sum_{t=1}^{q} U_{s,t}\,\mathrm{d}x_t \qquad \text{for} \quad s = 1, \ldots, p. \tag{5.4}$$

The substitution of (5.4) into the left-hand side of (5.3) gives rise to a linear function of $\mathrm{d}x_1, \ldots, \mathrm{d}x_q$. Because the variables $x_1, \ldots, x_q$ are independent, there cannot be any functional relation among them. Thus, the coefficients of the linear function in $\mathrm{d}x_1, \ldots, \mathrm{d}x_q$ must all vanish. This leads to the following system of q linear first-order partial differential equations, called the Jacobian system of (5.1):

$$\Delta_t\phi \equiv \frac{\partial\phi}{\partial x_t} + \sum_{s=1}^{p} U_{s,t}\frac{\partial\phi}{\partial u_s} = 0 \qquad \text{for} \quad t = 1, \ldots, q. \tag{5.5}$$

It is easy to see [11] that the two equations (5.5) and (5.4) have exactly the same integrals, and that a function Ψ is a solution of (5.4) or (5.5) if and only if it is a function of $\phi_1, \ldots, \phi_p$. It is also clear that both equations (5.4) and (5.5) have exactly p functionally independent solutions. Equation (5.4) is called the adjoint system of (5.5). Conversely, any equation of the form (5.5) satisfies the conditions of integrability if and only if the commutators of the differential operators Δ_t (for $t = 1, \ldots, q$) all vanish [11, 21].

5.2. Applications

Suppose that the n-dimensional Lie algebra L is generated by $\{X_1, \ldots, X_n\}$ and that $\{x_1, \ldots, x_n\}$ is a coordinate system associated with this basis. We notice that under the identification of the symmetric space $\mathfrak{S}(L^*)$ with $\mathfrak{S}(L)$, the coordinates system in a given basis of L^* may be replaced by the coordinates system in the corresponding dual basis in L. Furthermore, it is customary to use the same notation for basis vectors and corresponding coordinates systems in the expression of the invariants.

By (2.2), the determining equations $\tilde{X}_i \cdot F = 0$, for $i = 1, \ldots n$, are given by the system of homogeneous linear first-order PDEs

$$\sum_j [x_i, x_j] \cdot \frac{\partial F}{\partial x_j} = 0 \qquad i = 1, \ldots, n \tag{5.6}$$

where we have set $[x_i, x_j] = \sum_k c_{ij}^k x_k$. Let $q = \mathrm{rank}(\mathcal{M}_L) = \mathrm{rank}([x_i, x_j])$. Then we can solve (5.6) for q of the variables $\frac{\partial F}{\partial x_j}$. This yields an equation of the form

$$\frac{\partial F}{\partial x_t} = -\sum_{s=q+1}^{n} U_{s,t}\frac{\partial F}{\partial x_s} \qquad t = 1, \ldots, q. \tag{5.7}$$

This equation represents the Jacobian system for the integral equations of (5.6). Indeed, by equation (2.3), equation (5.6) possesses exactly $n - q$ functionally independent invariants. It also determines the coefficients $U_{s,t}$ of the corresponding adjoint system, and thus the adjoint system itself, which is given by

$$\mathrm{d}u_s = \sum_{t=1}^{q} U_{s,t}\,\mathrm{d}x_t \qquad \text{for} \quad s = 1, \ldots, n-q \quad \text{and} \quad u_s = x_{q+s}. \tag{5.8}$$

Consequently, the determining equations given by (5.6) and which represent a system of linear first-order PDEs are equivalent to (5.8). Therefore, the latter system can be used for the

Table 1. Invariants of the semi-direct sum $L = S \oplus_\pi \mathcal{R}(m)$.

Dimension of $\mathcal{R}(m)$	Algebra	Invariants
1	$\mathfrak{sl}(2, F) \oplus \langle v_0 \rangle$	$I_1 = 4xy + h^2$ $I_2 = v_0$
2	$\mathfrak{sl}(2, F) \oplus_\pi \mathcal{R}(1)$	$I_1 = v_1^2 x + v_0 v_1 h - v_0^2 y$
3	$\mathfrak{sl}(2, F) \oplus_\pi \mathcal{R}(2)$	$I_1 = hv_1 + 2v_2 x - 2v_0 y$ $I_2 = v_1^2 - 4v_0 v_2$
4	$\mathfrak{sl}(2, F) \oplus_\pi \mathcal{R}(3)$	$I_1 = 2v_0 v_2^3 - 9v_0 v_1 v_2 v_3 + \frac{27}{2} v_0^2 v_3^2 + \frac{1}{2} v_1^2 v_2^2 + 2v_1^3 v_3$
5	$\mathfrak{sl}(2, F) \oplus_\pi \mathcal{R}(4)$	$I_1 = -12v_0 v_4 + 3v_1 v_3 - v_2^2$ $I_2 = 27v_0 v_3^2 - 9v_1 v_2 v_3 + 27v_1^2 v_4 - 72v_0 v_2 v_4 + 2v_2^3$

determination of the invariants. As already noted, this system involves in many instances, and in particular when the number of variables in the equations is relatively high, a more systematic algorithm than that needed for solving the system of PDEs (5.6) directly.

5.3. Examples

The method consisting of solving the system (5.8) rather than the equivalent system of PDEs (5.6) has been used in [10] for the determination of the invariants. Namely, the invariants of certain solvable Lie algebras of dimension 6, and those of the Lie algebra $sa(n, \mathbb{R})$ (for $n = 2, 3, 4$), where $sa(n, \mathbb{R})$ is the semi-direct sum of $\mathfrak{sl}(n, \mathbb{R})$ and the Abelian Lie algebra of dimension n. We apply the same method here for the Lie algebra $L = S \oplus_\pi \mathcal{R}$, where as in section 3 this notation represents the semi-direct sum of $S = \mathfrak{sl}(2, \mathbb{K})$, and the radical $\mathcal{R}$, and where the commutation relations of the $[S, \mathcal{R}]$-type are defined by the irreducible representation π.

We shall suppose that the n-dimensional Lie algebra L has a basis of the form given in theorem 2. Thus, $\mathfrak{sl}(2, \mathbb{K})$ is generated by $\{x, y, h\}$ and the radical $\mathcal{R}$ of dimension d is generated by $\{v_0, v_1, \dots, v_m\}$, where $m = d - 1$. It is clear from equation (5.6) that the determining equations are completely determined by the commutator table of L. In turn, this table is completely determined by equation (4.1) and part (b) of theorem 2. However, to derive the adjoint system (5.8), we also need the rank of the matrix of the commutator table, and this is given by theorem 3. Once the adjoint system is obtained, it can be solved using one of the methods available for solving systems of total differential equations, and Natani's method appears to be the most appropriate in this case.

Table 1 gives a list of invariants computed for $L = S \oplus_\pi \mathcal{R}(m)$ where the dimension d of the irreducible S-module $\mathcal{R}(m)$ of maximal weight $m = d - 1$ is up to 5, and where L has the usual basis of the form $\{x, y, h, v_0, v_1, \dots, v_m\}$. We give here an example of the procedure of calculations when $d = 3$. The matrix $\mathcal{M}_L$ of the commutator table has the form

$$\mathcal{M}_L = \begin{pmatrix} 0 & h & -2x & 0 & 2v_0 & v_1 \\ -h & 0 & 2y & v_1 & 2v_2 & 0 \\ 2x & -2y & 0 & 2v_0 & 0 & -2v_2 \\ 0 & -v_1 & -2v_0 & 0 & 0 & 0 \\ -2v_0 & -2v_2 & 0 & 0 & 0 & 0 \\ -v_1 & 0 & 2v_2 & 0 & 0 & 0 \end{pmatrix}. \tag{5.9}$$

The notation $[x_i, x_j] = \sum_k c_{ij}^k x_k$ used in equation (5.6) shows that the determining equations can be read off the commutator table. For instance, if we denote by F_u the partial derivative $\partial F/\partial u$ of a function F with respect to the variable u, the functions invariant under the subgroup generated by h are given by the third row of (5.9) as

$$\tilde{h} \cdot F \equiv 2xF_x + -2yF_y + 2v_0F_{v_0} - 2v_2F_{v_2} = 0.$$

However, given the matrix $\mathcal{M}_L$ of the commutator table, and knowing that it has rank 4, we only need to solve the equation $\mathcal{M}_L \cdot X = 0$, for four of the components in the vector

$$X = \left(F_x, F_y, F_h, F_{v_0}, F_{v_1}, F_{v_2}\right)$$

in terms of the remaining two others. This determines the Jacobian system and the coefficients $U_{s,t}$, and hence the adjoint system. More precisely, solving for four of the variables in terms of the two variables F_{v_1} and F_{v_2} determines the coefficients $U_{s,t}$, with v_1 and v_2 as dependent variables. These transformations yield after simplification the following system of two total differential equations.

$$(hv_0 + v_1x)\,\mathrm{d}v_1 = -2v_2v_0\,\mathrm{d}x + 2v_0^2\,\mathrm{d}y - v_1v_0\,\mathrm{d}h + 2(v_2x + v_0y)\,\mathrm{d}v_0$$

$$(hv_0 + v_1x)\,\mathrm{d}v_2 = -v_1v_2\,\mathrm{d}x + v_0v_1\,\mathrm{d}y - \frac{v_1^2}{2}\,\mathrm{d}h + (v_1y - hv_2)\,\mathrm{d}v_0.$$

Solving this system gives the invariants

$$I_1 = hv_1 + 2v_2x - 2v_0y$$

$$I_2 = v_1^2 - h^2v_1^2 - 4v_0v_2 - 4hv_1v_2x - 4v_2^2x^2 + 4hv_0v_1y + 8v_0v_2xy - 4v_0^2y^2.$$

After simplification, these invariants that we call again I_1 and I_2 are given by

$$I_1 = hv_1 + 2v_2x - 2v_0y \qquad I_2 = v_1^2 - 4v_0v_2.$$

The computation of the invariants becomes more and more impractical when the dimension of the radical is greater than 5. This is partly due to the complications that arise when solving the adjoint system, owing to the fast growing number of terms and variables that appear in the invariants.

6. Properties of the invariants

We notice that all the invariants of L computed for $\dim \mathcal{R}(m) = 1, \ldots, 5$ and given in table 1 are all polynomials as stipulated by lemma 1. Moreover, in all cases there is a fundamental system of invariants consisting of homogeneous polynomials. On the other hand, for $d \geqslant 4$, we realize that the invariants F depend only on the elements of the irreducible S-module $\mathcal{R}(m)$. That is, they are of the form $F = F(v_0, v_1 \ldots, v_m)$. The following result generalizes this observation.

Theorem 5. *Let $L = S \oplus_\pi \mathcal{R}$, where π is an irreducible representation of $S = \mathfrak{sl}(2, \mathbb{K})$ in the radical $\mathcal{R}$, and suppose that* $\dim \mathcal{R} \geqslant 4$.

(a) Every invariant of L has the form $F = F(v_0, \ldots, v_m)$. That is, F does not depend on the variables x, y and h associated with $\mathfrak{sl}(2, \mathbb{K})$.

(b) The invariants of L are all completely determined by the $[S, \mathcal{R}]$-type commutation relations alone.

Proof. Let $\tilde{V}_0, \tilde{V}_1$ and $\tilde{V}_2$ be the infinitesimal generators corresponding to the basis elements v_0, v_1 and v_2. By the determining equations (2.2), any invariant F must satisfy the system of equations $\tilde{V}_0 \cdot F = \tilde{V}_1 \cdot F = \tilde{V}_2 \cdot F = 0$. Since $\mathcal{R}$ is Abelian according to an earlier remark, this system corresponds to a linear equation of the form $A \cdot Z = 0$, where Z is the vector (F_x, F_y, F_h) and where A is the submatrix of the commutator matrix $\mathcal{M}_L$ located between positions (4, 1) and (6, 3). The determinant of A is $(-2+3m-m^2)v_1^3 + (-6m+3m^2)v_0v_1v_2 - 3m^2v_0^2v_3$, which is clearly nonzero, and this proves the first part of the theorem.

For part (b), we notice that because of the condition $F_x = F_y = F_h = 0$ just proven in the first part above, and the fact that $\mathcal{R}$ is Abelian, the infinitesimal generators corresponding to the basis elements of $\mathcal{R}$ all reduce to zero. On the other hand, by equation (2.1) the infinitesimal generator $\tilde{E}_i$ corresponding to a basis element $E_i \in \{x, y, h\}$ reduces to

$$\tilde{E}_i = \sum_{j=0}^{m} [e_i, v_j] \frac{\partial}{\partial v_j}$$

where e_i is the corresponding coordinate for E_i. This proves the assertion and completes the proof of the theorem. $\square$

Invariant functions that depend only on the elements of a particular subalgebra occur frequently in the study of Lie algebras. Theorem 5 represents only a particular case of a more general framework in which such functions usually occur. Suppose that the finite-dimensional Lie algebra $\mathfrak{M} = L_1 \dot{+} L_2$ is a vector space direct sum of the subspace L_1 and the Abelian subalgebra L_2. Let $\{X_1, \ldots, X_t\}$ be a basis of L_2 and extend it to a basis $\{X_1, \ldots, X_t, X_{t+1}, \ldots, X_s\}$ of $\mathfrak{M}$. Denote by B the matrix $([x_i, x_j])_{\substack{i=1,\ldots,t \\ j=t+1,\ldots,s}}$, in which $x_1, \ldots, x_s$ is a coordinate system in the given basis of $\mathfrak{M}$. We have the following generalization of theorem 5.

Theorem 6. *Suppose that* $\dim L_2 \geqslant \dim L_1$ *and that the matrix* B *is of maximal rank.*

(a) Every invariant of $\mathfrak{M}$ *is of the form* $F = F(x_1, \ldots, x_t)$. *That is,* F *depends only on the elements of the Abelian subalgebra* L_2.

(b) All invariants of $\mathfrak{M}$ *are completely determined by the* $[L_1, L_2]$*-type commutation relations alone.*

Proof. Let $\tilde{X}_i$ be the infinitesimal generator of the coadjoint action corresponding to X_i. For every invariant F, equation (2.2) implies in particular that $\tilde{X}_i \cdot F = 0$ (for $i = 1, \ldots, t$). The corresponding system of PDEs can be written as a system of linear equations of the form $B \cdot Z = 0$, where Z is the vector $\left(F_{x_{t+1}}, \ldots, F_{x_s}\right)$. The condition $\dim L_2 \geqslant \dim L_1$ means that $s - t \leqslant t$ and this ensures that when B has maximal rank, it contains an invertible submatrix so that the linear system $B \cdot Z = 0$ implies $Z = 0$. The rest of the proof is similar to that given for theorem 5. $\square$

Example 1. Take $\mathfrak{M} = L_1 \dot{+} L_2$ to be a solvable and non-nilpotent Lie algebra having an Abelian nilradical L_2. By a result of [22, 23], we have $\dim L_2 \geqslant \dim \mathfrak{M}/2$. Furthermore, we showed in [23] that the corresponding matrix B has maximal rank. It follows that $\mathfrak{M}$ always satisfies the hypothesis of the theorem and thus the invariants of $\mathfrak{M}$ depend only on the elements of the nilradical, and they are completely determined by the $[L_1, L_2]$-type commutation relations alone. (This is theorem 3 and corollary 1 of [23].) The invariants of solvable non-nilpotent Lie algebras of dimension 6 over $\mathbb{R}$ having Abelian nilradicals are computed in [4]. None of them has a fundamental set consisting of polynomials. Moreover, they usually involve logarithms and functions in arctan.

Remark. Theorem 6 greatly simplifies the determination of the invariants, by reducing at least by half the number of equations in the system of determining equations given by (2.2), and by reducing by $\dim L_1$ the number of independent variables in these equations.

We showed that a semi-direct sum of Lie algebras is perfect when the representation associated with it does not possess a copy of the trivial representation. In such a case, semi-direct sums of Lie algebras always have a fundamental set consisting of polynomial invariants. We now make use of table 1 to show that despite these facts, and contrary to the case of semisimple Lie algebras, the number of their invariants is not the same as the dimension of the Cartan subalgebra. Indeed, when $\dim \mathcal{R} = 2$, the Lie algebra $L = \mathfrak{sl}(2, \mathbb{K}) \oplus_\pi \mathcal{R}(2)$ has only one invariant. However, it is easy to see that L is split over the field $\mathbb{K}$ of characteristic zero, and has Cartan subalgebra $\mathbb{K}h \oplus \mathcal{R}(2)$, which has dimension 3.

7. Conclusion

In this paper, we considered a semi-direct sum of Lie algebras of the form $L = S \oplus_\pi \mathcal{R}$, where π is a representation of the semisimple Lie algebra S in the radical $\mathcal{R}$ that defines the $[S, \mathcal{R}]$-type commutation relations. We showed that L is perfect when π does not possess a copy of the trivial representation and that this condition is equivalent to the requirement that $\pi(S)R = R$. In this case, L has a fundamental set of invariants consisting of polynomials (lemma 1). In particular, L has this property when π is irreducible (theorem 1). The number of invariants is given in theorem 4, when $S = \mathfrak{sl}(2, \mathbb{K})$. Using a method of total differential equations, we were able to determine the invariants when the dimension of the radical is up to 5 (table 1), and to derive a theorem on certain properties of these invariants (theorem 5), as well as a generalization of this theorem (theorem 6). Finally, we showed that, although the Lie algebras we considered have a fundamental set of invariants consisting of polynomials, the cardinality of this set is generally not equal to the dimension of the Cartan subalgebra, as it is in the well-known case of semisimple Lie algebras.

References

[1] Abellanas L and Martinez Alonso L 1975 A general setting for Casimir invariants *J. Math. Phys.* **16** 1580
[2] Patera J, Sharp R T, Winternitz P and Zassenhaus H 1976 Invariants of real low dimension Lie algebras *J. Math. Phys.* **17** 986
[3] Perroud M 1993 The fundamental invariants of inhomogeneous classical groups *J. Math. Phys.* **24** 1344
[4] Ndogmo J C 2000 Invariants of solvable Lie algebras of dimension six *J. Phys. A: Math. Gen.* **33** 2273
[5] Campoamor-Stursberg R 2003 Non-semisimple Lie algebras with Levi factor $so(3)$, $\mathfrak{sl}(2, \mathbb{R})$ and their invariants *J. Phys. A: Math. Gen.* **36** 1357
[6] Wigner E P 1939 *Ann. Math.* **40** 149
[7] Arima A and Iachello F 1975 *Phys. Rev. Lett.* **35** 1069
[8] Racah G 1950 Sulla caratterizzazione delle rappresentazioni irriducibli dei gruppi semisemplici di Lie (Italian) *Atti Accad. Naz. Lincei. Ren. Cl. Sci. Fis. Mat. Nat.* **8** 108
[9] Casimir H 1931 Uber die Konsttruktion einer zu den irreduziblen darstellungen halbeinfacher kontinuierlicher Gruppen geh origen Differentialgleichung *Proc. R. Acad. Amsterdam* **34** 844
[10] Pecina-Cruz J N 1994 An algorithm to calculate the invariants of any Lie algebra *J. Math. Phys.* **35** 3146
[11] Forsyth A R 1890 *Theory of Differential Equations: Part 1* (New York: Dover)
[12] Fomenko A T and Trofimov V V 1988 *Integrable Systems on Lie Algebras and Symmetric Spaces* (New York: Gordon and Breach)
[13] Beltrametti E G and Blasi A 1966 On the number of Casimir operators associated with any Lie group *Phys. Lett.* **20** 62
[14] Humphreys J E 1972 *Introduction to Lie Algebras and Representation Theory* (New York: Springer)
[15] Dixmier J 1974 *Enveloping Algebras* (Paris: Gauthier-Villars)

[16] Rand D, Winternitz P and Zassenhaus H 1988 On the identification of a Lie algebra given by its structure constants: I. Direct decomposition, Levi decompositions, and nilradicals *Linear Algebra Appl.* **109** 197

[17] Bernat P 1962 Sur le corps des quotients de l'algèbre enveloppante d'une algèbre de Lie *C. R. Acad. Sci. Paris* **254** 1712

[18] Campoamor-Stursberg R 2002 Invariants of solvable rigid Lie algebras up to dimension 8 *J. Phys. A: Math. Gen.* **35** 6293

[19] Sagle A A and Walde R E 1973 *Introduction to Lie Groups and Lie Algebras* (New York: Academic)

[20] Gantmacher F R 1960 *Matrix Theory* (New York: Chelsea)

[21] Goursat E 1917 *Mathematical Analysis: Part II* (Boston: Ginn)

[22] Mubarakzyanov G M 1966 Some Problems about solvable Lie algebras *Izv. Vyssh. Uchebn. Zaved. Mat.* **6** 95

[23] Ndogmo J C 2000 Properties of the invariants of solvable Lie algebras *Can. Math. Bull.* **43** 459